Chapter 5 Summary

Let $\{\lambda_1, \ldots, \lambda_k\}$ be the eigenvalues of an $n \times n$ matrix A.

- $\mathbb{R}^n$ has a basis consisting of eigenvectors of A if and only if all λ_j are real and $\dim E_{\lambda_j} = m_{\lambda_j}$ for every λ_j.
- If any λ_j is not real, then $\mathbb{R}^n$ does not have a basis of eigenvectors of A, but $\mathbb{C}^n$ might.
- $\mathbb{C}^n$ has a basis consisting of eigenvectors of A if and only if $\dim E_{\lambda_j} = m_{\lambda_j}$ for every λ_j after extending scalars to $\mathbb{C}$.
- If either $\mathbb{R}^n$ or $\mathbb{C}^n$ has a basis of eigenvectors of A, then $D = P^{-1}AP$, where P is the $n \times n$ matrix whose columns are the eigenvectors of A and D is the diagonal matrix of eigenvalues of A, listed with appropriate multiplicity.

Section 5.5: Jordan Blocks

Even if an $n \times n$ matrix A is not diagonalizable, we can always find a nicer 'canonical form' for A.

Let $\lambda \in \mathbb{C}$, and let k be a positive integer. The **Jordan block** of size k with eigenvalue λ, denoted $J_k(\lambda)$, is the $k \times k$ matrix with λ along the main diagonal and ones along the 'first superdiagonal':

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Any matrix which has a series of Jordan blocks along its diagonal, and zeros everywhere else, will be called a **Jordan Canonical Form**. The eigenvalues and sizes of the Jordan blocks occurring may be different. Thus, a Jordan Canonical Form looks like

$$\begin{pmatrix} J_{k_1}(\lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & J_{k_2}(\lambda_2) & 0 & \cdots & 0 \\ 0 & 0 & J_{k_3}(\lambda_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{k_m}(\lambda_m) \end{pmatrix},$$

where each 0 represents a zero matrix of the appropriate size.

Theorem: Given any matrix $A \in M_n(\mathbb{R})$, there is a Jordan Canonical Form J and an invertible $n \times n$ matrix P with

$$J = P^{-1}AP.$$

In the case that A does not have real eigenvalues, we will have to extend scalars to find J and P; that is, $J \in M_n(\mathbb{C})$ and $P \in M_n(\mathbb{C})$.

The matrix J is unique up to rearrangement of the Jordan blocks involved.

If A is diagonalizable, then J will be a diagonal matrix, i.e. each Jordan block involved will be a 1×1 matrix.

We will give examples of the Jordan Canonical Form for 2×2 and 3×3 matrices in class. These will be useful for solving certain types of linear systems of differential equations.